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THE LIMITING DISTRIBUTION FOR THE INFINITELY DEEP DAM WITH A MARKOVIAN INPUT

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The paper outlines a case for taking greater interest in the bottomless, or infinitely deep, dam model in Hydrology. It then shows that for such a model with unit withdrawals and an ergodic Markov chain input process the limiting distribution of depletion, when this exists, is a zero modified geometric distribution. This result generalises the well known result for independent inputs. The technical conditions required for the proof are satisfied for finite state space input processes and are shown to be satisfied by certain infinite state space input processes. These include as special cases examples which have a negative binomial limiting input distribution.

Finite dam	R-recurrence
infinitely high dam	branching process
infinitely deep dam	

1. Introduction

A much analysed model for a water storage dam is the left-continuous, integer valued, random walk, $\mathcal{U} = \{U_n\}$, moving between impenetrable barriers at zero and $K-1$ and defined by

$$U_{n+1} = [((U_n + X_n) \wedge K) - 1]^+. \tag{1}$$

In this representation K is the capacity of the dam, X_n is the inflow during the interval $(n, n+1)$ at the end of which a unit amount of water is removed, if it is available, and U_{n+1} is the content at $n+1$ after the outflow at $n+1$.

A problem of some interest to the hydrologist is that of determining the limiting content distribution $\pi_{\mathcal{U}}$. This will enable him to make preliminary estimates of the reliability of a proposed dam and to make rough comparisons between possible sizes and sites. When $\{X_n\}$ is a sequence of independent random variables with the probability generating function (p.g.f.) $f(\cdot)$, $0 < f(0) < 1$, this problem can be attacked by solving a system of K linear equations for $\pi_{\mathcal{U}}$. However, this is likely to be quite difficult if K is large and analytic results are available in only a few cases.

An approximate solution may be obtained by letting $K = \infty$, that is, considering the corresponding infinitely high dam process $\mathscr{V}$. If $\rho = f'(1-) < 1$, the limiting distribution exists and has the p.g.f.

$$\Pi_{\mathscr{V}}(s) = \sum_{j \geq 0} \pi_{\mathscr{V}}(j) s^j = \frac{(1-\rho)(1-s)}{f(s)-s}. \tag{2}$$

Even in this case there are few analytical results available and, more importantly, in practice $\rho > 1$ [6]; the draft is typically in the range of 50–80% of the mean input. The restriction $\rho < 1$ can be overcome by use of the ratio theorem stating that if $u(j)$ is the coefficient of s^j in $(1-s)/(f(s)-s)$, then

$$\pi_{\mathscr{U}}(j) = u(j) \Big/ \sum_{0 \leq i \leq K-1} u(i).$$

However, the computational difficulties associated with (2) still obtain.

In practice the inflow process does not have independent increments, the most deeply studied generalisation being that of a Markov dependent inflow process [1, 7, 13]. In this case analogues of the results mentioned above exist but the computational problems are far more difficult. Thus very few explicit results are known and these are sometimes trivial, for example, inflow processes whose state spaces have cardinality three.

We propose that it is more sensible to use an approximation based on the infinitely deep dam where the depletion from the maximum possible level, $K-1$, is represented by the process $\mathscr{W} = \{W_n\}$ given by

$$W_{n+1} = (W_n + 1 - X_n)^+, \tag{3}$$

a right-continuous random walk. If the inflow process is an independent sequence and $\rho > 1$, then, as is well-known, the limiting-stationary distribution π_W is given by

$$\pi_W(j) = (1-q)q^j \tag{4}$$

where q is the least positive root of $f(s) = s$. Moreover there are useful methods for the approximate determination of q; see [14]. The relation (4) yields the approximation

$$\pi_{\mathscr{U}}(j) \simeq (1-q)q^{K-1-j}/(1-q^K) \quad (j = 0, \ldots, K-1), \tag{5}$$

which is similar to approximations based on Wald's identity; see [11].

If the reliability specification of a proposed dam is that the probability of supply be P, then it is not difficult to show that this can be achieved by choosing [6, 12]

$$K \simeq 1 + (\log P)/(\log q). \tag{6}$$

However if it were the case that π_W has a simple form for more general input processes then approximations similar to those at (5) and (6) will still be available. It has been shown that the queue length process imbedded at arrival times for certain G/M/1 models has a limiting zero modified geometric distribution. This has been shown, for example, by Finch and Pearce [3] in the case where inter-arrival times form a moving average process and by Tin [16] for the case of a rather special Markov chain (a continuous branching process allowing immigration which has a gamma limiting distribution). Furthermore it has been conjectured by Finch [2] that this is a general phenomenon.

While this conjecture appears to be difficult to verify, it is our purpose in the next section to show that it is valid for a Markov chain input process, subject, of course, to some mild technical conditions. These are satisfied by any finite irreducible aperiodic Markov chain. In Section 3 we shall show that our conditions are also satisfied by the simple branching process allowing immigration, referred to as a linearly regressive process in this context, and also by a related chain. We shall also exhibit specific results for some special examples of the linearly regressive input process; these were first introduced in [13].

We finally conclude this introduction by pointing out that the very early, and for many years canonical work on estimation of dam size through the specification of a failure probability was based on an infinitely deep dam model [4]. A more detailed discussion of the relevance of the infinitely deep dam model in Hydrology can be found in [12].

The cases of seasonally varying inflows and non-unit outflows will be considered elsewhere.

2. The principal result

Assumption I. *$\mathscr{X}=\{X_n\}$ is a Markov chain having a positive recurrent state space $\mathscr{S}=\{0,1,\ldots,N\}$ $(N<\infty)$ or $\{0,1,\ldots\}$.*

Denote $\mathscr{X}$'s transition matrix by $\mathscr{P}=[p_{ij}]$ and its limiting stationary distribution by $\{u(j): j\in\mathscr{S}\}$. For the present we consider the model $\mathscr{W}$ defined by (3). The sequence $\{W_n, X_n\}$ is a Markov chain which is easily seen to have an irreducible aperiodic state space which, moreover, is positive recurrent if $\nu=\sum ju(j)>1$. We shall exhibit a sequence $\{c(v,j)\}$ which is a bivariate distribution and is invariant for the transition probabilities of $\{W_n, X_n\}$, and hence is the limiting-stationary distribution of this process. It then follows that

$$\pi(v)=\sum_{j\in\mathscr{S}} c(v,j)$$

defines the limiting distribution of $\{W_n\}$

Let $z_{ij}=s^i p_{ij}$ $(0<s\leq 1; i,j\in\mathscr{S})$ and $[z_{ij}^{(n)}]=[z_{ij}]^n$. It follows from the general theory of non-negative matrices [15] that

$$\lambda(s)=1/R(s)=\lim_{n\to\infty}(z_{jj}^{(n)})^{1/n}$$

exists and is independent of j. Furthermore $\lambda(s)$ is superconvex and hence continuous [5], and it is non-decreasing.

Assumption II. *The equation*

$$\lambda(s) = s$$

has a solution q in $(0, 1)$.

Convexity of $\lambda(\cdot)$ implies that q is the only solution in $(0, 1)$. In most circumstances we expect that $\lambda(0+)>0$, $\lambda(1-)=1$ and that $\lambda'(1-)$ exists and exceeds unity. Under these circumstances Assumption II is indeed fulfilled. We now fix $s = q$.

Assumption III. *The matrix $[z_{ij}]$ is R-recurrent with $R = R(q) = q^{-1}$.*

Denote the R-invariant measure by $\{a(j)\}$; it satisfies the equations

$$qa(j) = \sum_{i\in\mathscr{S}} a(i)q^i p_{ij} \quad (j \in \mathscr{S}) \tag{7}$$

and $a(j)>0$ $(j \in \mathscr{S})$.

Assumption IV. $\sum a(j)<\infty$.

Application of Fubini's theorem shows that Assumption IV is equivalent to assuming

$$\sum a(j)q^j < \infty,$$

a condition which may be easier to check. Finally we note that we can, and shall, normalise $\{a(j)\}$ so that $\sum_{j\in\mathscr{S}} a(j) = 1$.

Theorem 1. *If Assumptions* I–IV *are satisfied, then the limiting-stationary distribution of $\{W_n, X_n\}$ is given by*

$$c(v, j) = (1-q)(u(0)a(j)/a(0))q^v \quad (j \in \mathscr{S}, v \geq 1),$$

$$c(0, j) = u(j) - (u(0)/a(0))qa(j) \quad (j \in \mathscr{S}).$$

The limiting distribution of $\{W_n\}$ is given by

$$\pi(v) = (1-q)(u(0)/a(0))q^v \quad (v \geq 1),$$

$$\pi(0) = 1 - qu(0)/a(0).$$

Proof. We first show that $c(0, j)>0$. It then follows that $\{c(v, j)\}$ is indeed a probability distribution. Let $w(j) = u(j)/u(0)$, $\alpha(j) = a(j)/a(0)$ and $d(j) = w(j) - q\alpha(j)$. The vector $\boldsymbol{d} = (d(0), d(1), \ldots)$ satisfies

$$\boldsymbol{d} = \boldsymbol{d}\mathscr{P} + \boldsymbol{t} \tag{8}$$

where $t(j)=(\boldsymbol{t})_j=\sum_{i\in\mathcal{S}}\alpha(i)(q-q^i)p_{ij}$. Let $\mathcal{R}$ be the matrix with $\mathcal{R}_{0j}=p_{0j}$, $\mathcal{P}_{ij}=0$ $(i\neq 0)$ and set $Q=\mathcal{P}-\mathcal{R}$. Then (8) can be written as

$$\boldsymbol{d}=\boldsymbol{d}Q+\boldsymbol{\tau} \tag{9}$$

where $\boldsymbol{\tau}=\boldsymbol{d}\mathcal{R}+\boldsymbol{t}$ and

$$0<\tau(j)=\sum_{i\neq 0}\alpha(i)(q-q^i)p_{ij}<\infty.$$

Let $[q_{ij}^{(n)}]=Q^n$; eq. (9) yields

$$d(j)=\sum_{i\in\mathcal{S}}w(j)q_{ij}^{(n)}-q\sum_{i\in\mathcal{S}}\alpha(i)q_{ij}^{(n)}+\sum_{i\in\mathcal{S}}\tau(i)(\delta_{ij}+q_{ij}^{(1)}+\cdots+q_{ij}^{(n)}). \tag{10}$$

On observing that $q_{ij}^{(n)}\to 0$, it follows upon using dominated convergence, that the first and second sums on the right hand side of (10)$\to 0$ and hence that $d(j)>0$. We have used Assumption IV in a number of places. Conversely, if $c(0,j)>0$ it follows that Assumption IV holds.

Let $p(v,j\mid u,i)=P(W_{n+1}=v, X_{n+1}=j\mid W_n=u, X_n=i)$. Clearly

$$p(v,j\mid u,i)=p_{ij}\delta_{v,(u-i+1)^+}.$$

The proof will be complete once we show that

$$\sum_{i,u}c(u,i)p(v,j\mid u,i)=c(v,j). \tag{11}$$

The left hand side is

$$(1-q)(u(0)/a(0))\sum_{i,u}a(i)q^u p_{ij}\delta_{v,(u-i+1)^+}$$
$$+\sum_{i\geq 0}[u(i)-qu(0)a(i)/a(0)-(1-q)u(0)a(i)/a(0)]p_{ij}\delta_{v,(1-i)^+}. \tag{12}$$

If $v\geq 1$, $\delta_{v,(u-i+1)^+}=\delta_{v,(u-i+1)}$ and hence the first sum is

$$(1-q)(u(0)/a(0))\sum a(i)q^{i-1+v}p_{ij}=c(v,j),$$

where we have used (7). Since $1-i\leq 1$ all terms in the second sum are zero if $v\geq 2$. If $v=1$ then $\delta_{v,(1-i)^+}=0$ unless $i=0$, but the first term in the second sum is always zero. Thus the system (11) is satisfied if $v\geq 1$.

When $v=0$, $\delta_{v,(u-i+1)^+}=1$ iff $u\leq i-1$ and $i\geq 1$. Thus the first sum at (12) is

$$(1-q)(u(0)/a(0))\sum_{i\geq 1}\sum_{u=0}^{i-1}a(i)q^u p_{ij}=(u(0)/a(0))\Big(\sum_{i\in\mathcal{S}}a(i)p_{ij}-qa(j)\Big)$$

Since $\delta_{0,(1-i)^+}=1$ iff $i\geq 1$, the second sum at (12) is

$$\sum_{i\in\mathcal{S}}(u(i)-(u(0)/a(0))a(i))p_{ij}=u(j)-(u(0)/a(0))\sum_{i\in\mathcal{S}}a(i)p_{ij}$$

and hence (11) is satisfied for $v=0$.

Let $u=u(0)$ and $a=a(0)$. Theorem 1 shows that the limiting depletion is known once we obtain u, a and q. These can be numerically determined without undue difficulty when $N<\infty$, but may be more difficult to obtain when $N=\infty$. However it is still the case that this problem is far more tractable than the corresponding problem for the infinitely high dam.

Under Assumptions I–IV, the approximation corresponding to (5) is

$$\pi_{\mathcal{U}}(j)\simeq\pi(K-1-j)/(1-uq^K/a)$$

and (6) is replaced by

$$K\simeq 1+(\log aP/u))/(\log q).$$

4. Examples

Pakes [9] analysed the model $\mathcal{V}$ when $\mathcal{X}$ was the linearly regressive process. For this input process p_{ij} is the coefficient of s^j in $h(s)(f(s))^i$ where h and f are p.g.f.'s satisfying $0<h(0), f(0)<1$. This process contains as special cases some parametric models first investigated by Phatarfod and Mardia [13] and it is fairly tractable with respect to the implementation of Theorem 1. Thus we shall now assume that $\mathcal{X}$ is such a process. There are good hydrological reasons for making this choice. If $\alpha=f'(1-)$, $\beta=h'(1-)<\infty$, then $\mathcal{X}$ satisfies the linear regression property

$$E(X_{n+1}\mid X_n)=\beta+\alpha X_n$$

and a similar relation is satisfied by $\operatorname{var}(X_{n+1}\mid X_n)$, provided $f''(1-), h''(1-)<\infty$. These relations imply a tendency for greater persistence of low inflows than high inflows, that is, a drought is more likely to continue than is a flood; this is an empirically observed phenomenon.

If $\alpha<1$, $\mathcal{X}$ is positive recurrent and the same can also be true under rather exceptional circumstances when $\alpha=1$. We shall assume $\alpha<1$. The p.g.f. $U(s)=\sum_{j\geq 0}u(j)s^j$ is given by

$$U(s)=\prod_{n=0}^{\infty}h(f_n(s))$$

where $f_0(s)=s$, $f_{n+1}(s)=f(f_n(s))$ $(n=0,1,\ldots)$. Furthermore $U'(1-)=\beta/(1-\alpha)$ and hence $\{W_n, X_n\}$ has a limiting distribution if $\alpha+\beta>1$.

The spectral properties of the matrix $[w_{ij}]=[p_{ij}s^i]$ were obtained in [9]. Since $z_{ij}^{(n)}=s^{i-j}w_{ij}^{(n)}$, the spectral properties of $[z_{ij}]$ follow from those of $[w_{ij}]$. In particular the two matrices have the same convergence norm and R-classification. Let $g(t)$ be the solution of $s=tf(s)$; $g(t)$ is a p.g.f. Pakes [9] proved that the convergence norm of $[w_{ij}]$ is $1/V(s)$ when $V(s)=h(g(s))$. This is a p.g.f., $V'(1-)=\beta/(1-\alpha)$, $V(0)=h(0)>0$ and hence Assumption II is satisfied. For $s=q$ it also follows from the results in [9] that

$$w_{ij}^{(n)}\sim(g(q)/q)^i\mu_j(q)q^n$$

where $\{\mu_j(q)\}$ is q^{-1}-invariant for $[w_{ij}]$ and is given by

$$\mu(q,t)=:\sum_{j\geq 0}\mu_j(q)t^j=\prod_{m=0}^{\infty}[(h(\gamma_m(t))/q] \tag{13}$$

and $\gamma_0(t)=qt$, $\gamma_{n+1}(t)=qf(\gamma_n(t))$. It follows that $[z_{ij}]$ is q^{-1}-positive and that $a(j)$ is proportional to $\mu_j(q)q^{-j}$. Thus each assumption of Theorem 1 will be fulfilled once it is shown that $\sum\mu_j(q)q^{-j}<\infty$ or equivalently that $\sum\mu_j(q)<\infty$. But this is known to be the case; [9, p. 330]. We then have

$$a(j)=\mu_j(q)q^{-j}/\mu(q,q^{-1})$$

and

$$\mu(q,q^{-1})=\prod_{m=0}^{\infty}[h(\delta_m)/q]$$

where $\delta_0=1$, $\delta_{n+1}=qf(\delta_n)$. Furthermore

$$\mu(q,0)=\prod_{m=0}^{\infty}[h(\Delta_m)/q]$$

where $\{\Delta_n\}$ is defined as is $\{\delta_n\}$ but with $\Delta_0=0$. The desired quantity is $a(0)$, given by

$$a(0)=\prod_{m=0}^{\infty}\frac{h(\Delta_m)}{h(\delta_m)}.$$

A special case [13] is that where

$$h(s)=[1+a(1-\rho)-a(1-\rho)s]^{-k},$$

$$f(s)=\frac{(1+a)(1-\rho)-(a(1-\rho)-\rho)s}{1+a(1-\rho)-a(1-\rho)s},$$

$0<\rho<1$, $k>0$ and we assume that $\alpha/(1-\beta)=ak>1$. For this inflow process the limiting distribution is the negative binomial whose p.g.f. is $U(s)=(1+a-as)^{-k}$. The function $g(\cdot)$ satisfies a quadratic equation whose solution shows that $V(s)=\lambda_1^{-k}(s)$ where

$$\lambda_1(s)=\tfrac{1}{2}\{1+a(1-\rho)+(a(1-\rho)-\rho)s \\ -[(1+a(1-\rho)-(a(1-\rho)-\rho)s)^2-4\rho s]^{1/2}\}.$$

Thus the equation for q is not usually exactly soluble, but it is amenable to numerical methods. Setting

$$[1+a(1-\rho)-a(1-\rho)\gamma_n(t)]^{-1}=D_n/D_{n+1} \quad (n=0,1,\ldots)$$

reduces the recurrence relation for the γ_n to the difference equation

$$D_{n+2}-[1+a(1-\rho)-(a(1-\rho)-\rho)q]D_{n+1}+\rho q D_n=0$$

where $D_0=1$ and $D_1=1+a(1-\rho)-a(1-\rho)qt$, which follow from the form of γ_1. It follows that

$$D_n=E\lambda_1^n+F\lambda_2^n$$

where

$$E=(1+a(1-\rho)-a(1-\rho)qt-\lambda_2)/(\lambda_1-\lambda_2),$$
$$F=(1+a(1-\rho)-a(1-\rho)qt-\lambda_1)/(\lambda_2-\lambda_1),$$

$\lambda_1=\lambda_1(q)=q^{-1/k}$ and $\lambda_2=\rho q/\lambda_1$, being the other root of the quadratic equation satisfied by λ_1. Since $h(\gamma_n(t))=(D_n/D_{n+1})^k$, it follows from (13) that

$$\mu(q,t)=\lim_{n\to\infty} q^n D_{n+1}^{-k}=E^{-k}$$

and hence that

$$a(j)=\frac{(1-\lambda_2)^k}{(1+a(1-\rho)-\lambda_2)^{k+j}}\binom{-k}{j}(-aq(1-\rho))^j.$$

When $k=1$, E can be simplified since $q=(2\rho)^{-1}[-a(1-\rho)+(a^2(1-\rho)^2+4\rho)^{1/2}]$, and from $\lambda_2=\rho q^2$ we find that the numerator in E is $a(1-\rho)(1+q-qt)$. It follows that

$$a(j)=(1+q)^{-1}(q/(1+q))^j$$

and hence that

$$\pi(0)=1-\frac{1+q}{1+a} \quad\text{and}\quad \pi(j)=\frac{1-q^2}{1+a}q^{j-1} \quad (j\geq 1).$$

Substituting a by $-a$ and $-k$ by $r\in\mathbb{N}$ gives the results for an inflow process having a binomial limiting distribution.

We shall now consider the input process $\mathscr{X}$ defined by

$$\sum_{i\geq 0} s^i p_{ij}=H(s)(cf(s))^j \quad (j=0,1,\ldots) \tag{14}$$

where $0<c\leq 1$, $0<f(0)<1$, necessarily

$$H(s)=\frac{1-cf(s)}{1-s} \tag{15}$$

and $f(s)$ is a p.g.f. as above. The detailed properties of $\mathscr{X}$ are developed in [10]. Suffice it to say that this inflow process offers the possibility of an exact analysis of the transient depletion distribution. In addition if $c=1$ and $\alpha<\infty$, then $\mathscr{X}$ is asymptotically linearly regressive:

$$\mathbf{E}(X_{n+1} \mid X_n = i) \sim i/\alpha \qquad (i \to \infty). \tag{16}$$

This follows from a simple renewal theoretic argument.

Suppose that either $c<1$ or that $c=1$ and $\alpha>1$. The equation $s=cf(s)$ has a solution $\zeta \in (0,1)$ and hence from (14) and (15),

$$\sum_{i\geq 0} \zeta^i p_{ij} = \zeta^j.$$

Thus $\mathscr{X}$ has a geometric limiting distribution:

$$u(j) = (1-\zeta)\zeta^j.$$

Assume now that this limiting distribution exists and also that $\nu = \zeta/(1-\zeta) > 1$, that is, $\zeta > \frac{1}{2}$. Assumptions I–IV are satisfied. To see this let $z_{ij} = s^i p_{ij}$ and $Z_j^{(n)}(x) = \sum_{i\geq 0} x^i z_{ij}^{(n)}$. Then $Z_j^{(1)}(x) = H(xs)(cf(xs))^j$ and hence

$$Z_j^{(n+1)}(x) = H(xs) Z_j^{(n)}(cf(xs)). \tag{17}$$

Define $\gamma_n \equiv \gamma_n(x,s)$ through $\gamma_0 = xs$ and $\gamma_{n+1} = csf(\gamma_n)$. An induction argument shows that $\gamma_{n+1}(x,s) = \gamma_n(cf(xs), s)$; see for example [8]. Iteration of (17) yields

$$Z_j^{(n)}(x) = (\gamma_n/s)^j \prod_{m=0}^{n-1} H(\gamma_m).$$

If $0 \leq s < 1$ the sequence $\{\gamma_n\}$ converges to $g(s)$, the solution of $t = csf(t)$, and hence $g(0)=0$, $g(1-)=\zeta$ and $g(\cdot)$ has a power series expansion around zero which has non-negative coefficients. Finally, the convergence of $\{\gamma_n\}$ to $g(\cdot)$ is monotonic with a geometrically fast rate. The proof of these assertions follows from Lemmas 2 and 3 in [8]. Let $\lambda(s) = H(g(s))$. Arguing as in the proof of Lemma 1 in [9] it is easily shown that

$$Z_j^{(n)}(x)/(\lambda(s))^n \to (g(s)/s)^j \Gamma(x,s)$$

where $\Gamma(x,s) = \prod_{m=0}^{\infty} [H(\gamma_m)/\lambda(s)]$ converges uniformly with respect to $x \in [0,1]$. Moreover $\Gamma(x,s)$ has the power series expansion $\sum_{i=0}^{\infty} e_i(s) x^i$, say, whence

$$z_{ij}^{(n)} \sim (\lambda(s))^n e_i(s) \mu_j(s)$$

where $\mu_j(s) = (g(s)/s)^j$. These results show that $[z_{ij}]$ is $1/\lambda(s)$-recurrent and that $\{\mu_j(s)\}$ is an $1/\lambda(s)$-invariant measure.

It is easily checked that $\lambda(1-) = 1$, $\lambda'(1-) = \nu$ and that $\lambda(\cdot)$ is convex in $(0,1)$. Thus the equation $s = \lambda(s)$ has a root $q \in (0,1)$ and it readily follows that Assumptions I–IV are satisfied with

$$a(j) = (1-\delta)\delta^j \quad \text{where } \delta = g(q)/q < 1.$$

The last inequality results from the properties: $g(0)=0$, $g(1-)<1$ and $g(\cdot)$ is convex. Applying Theorem 1 we find that the dam deficit limiting distribution is

$$\pi(v)=(1-\zeta)q^{v} \quad (v\geqslant 1),$$
$$=1-q(1-\zeta)/(1-q) \quad (v=0)$$

and that of $\{W_n, X_n\}$ is

$$c(v,j)=(1-\zeta)(1-q)q^{v+j} \quad (v\geqslant 1),$$
$$=(1-\zeta)(\zeta^{j}-q^{j}) \quad (v=0).$$

In general $H(g(s))=(s-g(s))/s(1-g(s))$ and hence the equation determining q becomes

$$g(q)=q/(1+q). \tag{18}$$

We shall now consider the case where $c=1$, $f(s)=(1+\alpha-\alpha s)^{-1}$ and $\alpha>1$. It follows that $\zeta=\alpha^{-1}$ and hence $\nu>1$ iff $\alpha<2$. To find q first observe that for this case $H(s)=\alpha f(s)$ and hence that $\lambda(s)=\alpha g(s)/s$. Thus the equation determining q becomes $q^2=\alpha g(q)$ which with (18) yields

$$q^2+q-\alpha=0 \tag{19}$$

whence

$$q=((1+4\alpha)^{1/2}-1)/2.$$

The limiting distribution is now completely specified.

As we have mentioned in Section 1, when $\mathscr{X}$ is an independent sequence then $\mathscr{W}$ has a geometric limiting distribution. This can occur for the present input process, at least for special choices of the parameter α. The required condition is that $q=\zeta=\alpha^{-1}$ and hence, from (19), we need to satisfy

$$q^3+q^2-1=0.$$

This equation has a root close to 0·75488 and it follows, by continuity, that the required conditions can be met.

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